

Recursion Operators for a Class of Integrable Third-Order Evolution Equations

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Abstract

We consider $u_t = u^\alpha u_{xxx} + n(u)u_x u_{xx} + m(u)u_x^3 + r(u)u_{xx} + p(u)u_x^2 + q(u)u_x + s(u)$ with $\alpha = 0$ and $\alpha = 3$, for those functional forms of m , n , p , q , r , s for which the equation is integrable in the sense of an infinite number of Lie-Bäcklund symmetries. Local x - and t -independent recursion operators that generate these infinite sets of symmetries are obtained for the equations. A combination of potential forms, hodograph transformations and x -generalised hodograph transformations are applied to the obtained equations.

1 Introduction

A partial differential equation that admits a recursion operator which generates an infinite number of local x - and t -independent Lie-Bäcklund symmetries is called integrable [7, 8].

In the present paper we classify the $(1+1)$ -dimensional semilinear evolution equation of the form

$$u_t = u_{xxx} + n(u)u_x u_{xx} + m(u)u_x^3 + r(u)u_{xx} + p(u)u_x^2 + q(u)u_x + s(u), \quad (1.1)$$

as well as the quasilinear evolution equation of the form

$$u_t = u^3 u_{xxx} + n(u)u_x u_{xx} + m(u)u_x^3 + r(u)u_{xx} + p(u)u_x^2 + q(u)u_x + s(u), \quad (1.2)$$

where the C^3 -functions m , n , p , q , r , s are constrained by the existence of a local recursion operator of the form

$$R[u] = GD_x^2 + QD_x + H + \sum_j I_j D_x^{-1} J_j. \quad (1.3)$$

Here the functions G , Q , H , I_j , J_j depend on u and x -derivatives of u and are determined for (1.1) and (1.2) by the commutator relation

$$\left[L[u], R[u] \right] \varphi = \left(D_t R[u] \right) \varphi, \quad (1.4)$$

where $L[u]$ is the linear operator

$$L[u] = \frac{\partial F}{\partial u} + \frac{\partial F}{\partial u_x} D_x + \frac{\partial F}{\partial u_{xx}} D_x^2 + \cdots + \frac{\partial F}{\partial u_{x^n}} D_x^n, \quad (1.5)$$

F is the right-hand-side of (1.1) or (1.2) and $D_t R[u]$ calculates the explicit derivative with respect to t . For more details on recursion operators we refer to [7, 8].

In two recent papers [6, 5] we classified $(1+1)$ -dimensional autonomous evolution equations which can be transformed to linear evolution equations under the x -generalised hodograph transformation

$${}_n \mathbf{H} : \begin{cases} dX(x, t) = f_1(x, u)dx + f_2(x, u, u_x, u_{xx}, \dots, u_{x^{n-1}})dt \\ dT(x, t) = dt \\ U(X, T) = g(x) \end{cases} \quad (1.6)$$

with $n = 2, 3, \dots$, where

$$u_t \frac{\partial f_1}{\partial u} = \frac{\partial f_2}{\partial x} + u_x \frac{\partial f_2}{\partial u} + u_{xx} \frac{\partial f_2}{\partial u_x} + \cdots + u_{x^n} \frac{\partial f_2}{\partial u_{x^{n-1}}}. \quad (1.7)$$

We call these equations linearisable. For second-order equations the x -generalised hodograph transformation ${}_2 \mathbf{H}$ leads to eight linearisable equations [6]. We list here those linearisable equations which admit x - and t -independent local recursion operators [5]:

$$u_t = u_{xx} + \left(\frac{k''}{k'} + \alpha k' \right) u_x^2 + \beta u_x + \frac{\gamma}{k'} \quad (1.8)$$

$$R[u] = D_x + \left(\frac{k''}{k'} + \alpha k' \right) u_x \quad (1.9)$$

$$u_t = u_{xx} + \frac{k''}{k'} u_x^2 + \alpha k u_x + \beta u_x \quad (1.10)$$

$$R[u] = D_x + \frac{k''}{k'} u_x + \frac{\alpha}{2} k + \frac{\alpha}{2} u_x D_x^{-1} k' \quad (1.11)$$

$$u_t = k^2 u_{xx} + k^2 \frac{k''}{k'} u_x^2 + \alpha k^2 u_x + \beta u_x \quad (1.12)$$

$$R[u] = k D_x + k \frac{k''}{k'} u_x + \alpha k + \left(k^2 u_{xx} + k^2 \frac{k''}{k'} u_x^2 + \alpha k^2 u_x \right) D_x^{-1} \frac{k'}{k^2}. \quad (1.13)$$

Here α , β , γ are arbitrary constants and k is a nonconstant \mathcal{C}^3 function of u . In the present paper we exclude these linearisable cases already listed in [5].

The article is organized as follows: In Section 2 and Section 3 we classify the integrable equations of the form (1.1) and (1.2) with respect to their local recursion operators. The results are presented in Proposition 2.1 and Proposition 3.1. In all cases we give the next equations in the integrable hierarchies, which are all fifth-order equations. Several examples are given and all equations are presented in a more general form under the substitution $u \rightarrow k(u)$. In Section 4 we consider sequences of coordinate transformations for the integrable equations provided by Propositions 2.1 and 3.1, namely combinations of potential forms, hodograph transformations and x -generalised hodograph transformations. As mentioned above, we are interested in local recursion operators. However, we make one exception in Subsection 4.4, where we derive a nonlocal recursion operator for eq. (4.37), which is a linearisable third-order equation. This provides an interesting example of nonlocal recursion operators and its corresponding nonlocal symmetries and sets the stage for future investigations. In Section 5 we make some conclusions and list all equations for which we established recursion operators.

2 Recursion operators for the semilinear equation (1.1)

The recursion operator Ansatz (1.3) for (1.1) leads to

Proposition 2.1: *Evolution equations of the form (1.1), viz.*

$$u_t = u_{xxx} + n(u)u_x u_{xx} + m(u)u_x^3 + r(u)u_{xx} + p(u)u_x^2 + q(u)u_x + s(u)$$

which admit recursion operators of the form (1.3), viz.

$$R[u] = GD_x^2 + QD_x + H + \sum_j I_j D_x^{-1} J_j$$

and which are not linearisable by a x -generalised hodograph transformation, are exhausted by the following two cases:

I) *The equation*

$$\boxed{u_t = u_{xxx} + \lambda_1 u_x^3 + \lambda_2 u_x^2 + \lambda_3 u_x + \lambda_4} \quad (2.1)$$

with $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ arbitrary constants, admits the recursion operator

$$R[u] = D_x^2 + 2\lambda_1 u_x^2 + \frac{4\lambda_2}{3} u_x - 2\lambda_1 u_x D_x^{-1} u_{xx} - \frac{2\lambda_2}{3} D_x^{-1} u_{xx}. \quad (2.2)$$

II) *The equation*

$$\boxed{u_t = u_{xxx} + n(u)u_x u_{xx} + m(u)u_x^3 + q(u)u_x} \quad (2.3)$$

admits the recursion operator (1.3), where

$$\begin{aligned} Q &= \frac{2}{3}nu_x, & G &= 1 \\ H &= \frac{1}{3}nu_{xx} - \frac{1}{9}(3n' + n^2 - 18m)u_x^2 + \frac{2}{3}q \\ I &= u_x, \\ J &= \frac{2}{9}(3n' + n^2 - 9m)u_{xx} + \frac{1}{9}(3n'' + 2nn' - 9m')u_x^2 + \frac{1}{3}q' \end{aligned}$$

and m, n, p, q, s satisfy the following conditions:

$$q''' - nq'' - 3n'q' - \frac{2}{3}n^2q' + 8mq' = 0 \quad (2.4)$$

$$9m' - 3n'' - 6nm + \frac{2}{3}n^3 = 0. \quad (2.5)$$

To prove Proposition 2.1 one needs to verify commutator relation (1.4) for eq. (1.1) and the recursion operator Ansatz (1.3). This is done by direct calculations, so we do not show the details here.

In order to obtain the next equations in the integrable hierarchies of the equations in Proposition 2.1 we apply the recursion operators on the t -translation symmetries, i.e. $R[u]u_t$. For eq. (2.1) the next equation in the hierarchy is the fifth-order equation

$$\begin{aligned} u_t &= u_{xxxxx} + \lambda_2 \left(\frac{10}{3}u_x u_{xxx} + \frac{5}{3}u_{xx}^2 + \frac{10\lambda_2}{9}u_x^3 \right) + \lambda_3 (u_{xxx} + \lambda_1 u_x^3 + \lambda_2 u_x^2) \\ &+ \frac{2\lambda_4}{3}u_x + \lambda_1 \left(5u_x^2 u_{xxx} + 5u_x u_{xx}^2 + \frac{3\lambda_1}{2}u_x^5 + \frac{5\lambda_2}{2}u_x^4 \right). \end{aligned} \quad (2.6)$$

We note that eq. (2.1) of Proposition 2.1 contains the well known potential KdV equation (with $\lambda_1 = \lambda_4 = 0$) as well as the potential modified KdV equation (with $\lambda_2 = \lambda_4 = 0$). Equation (2.1) and its recursion operator has also been reported in [7].

By the substitution

$$u \rightarrow k(u), \quad k \in \mathcal{C}^3$$

eq. (2.1) takes the form

$$u_t = u_{xxx} + 3\frac{k''}{k'}u_x u_{xx} + \left(\frac{k'''}{k'} + \lambda_1(k')^2 \right) u_x^3 + \lambda_2 k' u_x^2 + \lambda_3 u_x + \frac{\lambda_4}{k'} \quad (2.7)$$

and admits the recursion operator

$$\begin{aligned} R[u] &= D_x^2 + 2\frac{k''}{k'}u_x D_x + \frac{k''}{k'}u_{xx} + \left(\frac{k'''}{k'} + 2\lambda_1(k')^2 \right) u_x^2 + \frac{4\lambda_2}{3}k' u_x \\ &- 2\lambda_1 u_x D_x^{-1} [(k')^2 u_{xx} + k' k'' u_x^2] - \frac{2}{3}\frac{\lambda_2}{k'} D_x^{-1} [(k')^2 u_{xx} + k' k'' u_x^2]. \end{aligned} \quad (2.8)$$

We now give some examples of equation (2.3) and its recursion operators of Case II in Proposition 2.1

Example 2.1: Let

$$m(u) = -\frac{1}{54}u$$

and make use of the substitution $u(x, t) = 3\tilde{u}(x, t)$. Then system (2.4) – (2.5) reduces to

$$n'' = 2n^3 + n\tilde{u} - \frac{1}{2} \quad (2.9)$$

$$q''' - 3nq'' - 9\left(n' + \frac{2}{3}n^2 + \frac{4}{9}\tilde{u}\right)q' = 0. \quad (2.10)$$

Here $' = d/d\tilde{u}$, etc. We note that (2.9) is the well known second Painlevé transcendent. Hence, system (2.9) – (2.10) is solvable in general. By Proposition 2.1 the equation

$$\boxed{\tilde{u}_t = \tilde{u}_{xxx} + 3n(\tilde{u})\tilde{u}_x\tilde{u}_{xx} - \frac{1}{2}\tilde{u}\tilde{u}_x^3 + q(\tilde{u})\tilde{u}_x} \quad (2.11)$$

where n and q satisfy (2.9) – (2.10), admits the recursion operator

$$\begin{aligned} R[\tilde{u}] &= D_x^2 + 2n\tilde{u}_xD_x + n\tilde{u}_{xx} - (n' + n^2 + \tilde{u})\tilde{u}_x^2 + \frac{2}{3}q \\ &\quad + \tilde{u}_xD_x^{-1} \left[2\left(n' + n^2 + \frac{1}{2}\tilde{u}\right)\tilde{u}_{xx} + \left(n'' + 2n'n + \frac{1}{2}\right)\tilde{u}_x^2 + \frac{1}{3}q' \right]. \end{aligned} \quad (2.12)$$

The next equation in the hierarchy is

$$\begin{aligned} \tilde{u}_t &= \tilde{u}_{xxxxx} + 5n\tilde{u}_x\tilde{u}_{xxx} + 10n\tilde{u}_{xx}\tilde{u}_{xxx} + 5\left(n' + n^2 - \frac{1}{2}\tilde{u}\right)\tilde{u}_x^2\tilde{u}_{xxx} \\ &\quad + 10\left(n' + n^2 - \frac{1}{4}\tilde{u}\right)\tilde{u}_x\tilde{u}_{xx}^2 + 5\left(nn' + n^3 - \frac{1}{2}\tilde{u}n - 1\right)\tilde{u}_x^3\tilde{u}_{xx} \\ &\quad + \frac{1}{2}\left(-(n')^2 + n^4 + \tilde{u}n^2 - 2n + \frac{3}{4}\tilde{u}^2\right)\tilde{u}_x^5 + \frac{5}{3}q\tilde{u}_{xxx} \\ &\quad + 5\left(\frac{2}{3}q' + nq\right)\tilde{u}_x\tilde{u}_{xx} + \left(\frac{5}{6}q'' + \frac{5}{2}nq' - \frac{1}{2}\tilde{u}q\right)\tilde{u}_x^3 + \frac{5}{6}q^2\tilde{u}_x, \end{aligned} \quad (2.13)$$

where the conditions on n and q satisfy (2.9) – (2.10).

Example 2.2: Let

$$m(u) = 0, \quad n(u) = 3u^{-1}$$

$$q(u) = \lambda_1 u^4 + \lambda_2 u^2 + \lambda_3, \quad \lambda_1, \lambda_2, \lambda_3 \in \mathfrak{R},$$

which satisfies system (2.4) – (2.5). Then (2.3) takes the form

$$\boxed{u_t = u_{xxx} + 3u^{-1}u_xu_{xx} + \lambda_1 u^4u_x + \lambda_2 u^2u_x + \lambda_3 u_x} \quad (2.14)$$

and by Proposition 2.1, eq. (2.14) admits the recursion operator

$$\begin{aligned} R[u] &= D_x^2 + 2u^{-1}u_x D_x + u^{-1}u_{xx} + \frac{2\lambda_1}{3}u^4 + \frac{2\lambda_2}{3}u^2 \\ &\quad + \frac{2}{3}u_x D_x^{-1} [2\lambda_1 u^3 + \lambda_2 u]. \end{aligned} \quad (2.15)$$

The next equation in the hierarchy is

$$\begin{aligned} u_t &= u_{xxxxx} + 5u^{-1}u_x u_{xxxx} + 10u^{-1}u_{xx} u_{xxx} + \frac{5\lambda_1}{3} (u^4 u_{xxx} + 11u^3 u_x u_{xx} + 12u^2 u_x^3) \\ &\quad + \frac{5\lambda_2}{3} (u^2 u_{xxx} + 7u u_x u_{xx} + 4u^3) + \lambda_3 (u_{xxx} + 3u^{-1}u_x u_{xx} + \lambda_1 u^4 u_x + \lambda_2 u^2 u_x) \\ &\quad + \frac{5\lambda_1^2}{6} u^8 u_x + \frac{5\lambda_1 \lambda_2}{3} u^6 u_x + \frac{5\lambda_2^2}{6} u^4 u_x. \end{aligned} \quad (2.16)$$

By the substitution

$$u \rightarrow k(u), \quad k \in \mathcal{C}^3,$$

eq. (2.14) takes the form

$$\boxed{u_t = u_{xxx} + 3 \left(\frac{k''}{k'} + \frac{k'}{k} \right) u_x u_{xx} + \left(\frac{k'''}{k'} + 3 \frac{k''}{k} \right) u_x^3 + \lambda_1 k^4 u_x + \lambda_2 k^2 u_x + \lambda_3 u_x} \quad (2.17)$$

so that system (2.4) – (2.5) admits the solution

$$\begin{aligned} n(u) &= 3 \left(\frac{k''}{k'} + \frac{k'}{k} \right), \quad m(u) = \frac{k'''}{k'} + 3 \frac{k''}{k} \\ q(u) &= \lambda_1 k^4 + \lambda_2 k^2 + \lambda_3. \end{aligned}$$

By Proposition 2.1, eq. (2.17) admits the recursion operator

$$\begin{aligned} R[u] &= D_x^2 + 2 \left(\frac{k''}{k'} + \frac{k'}{k} \right) u_x D_x + \left(\frac{k''}{k'} + \frac{k'}{k} \right) u_{xx} + \left(\frac{k'''}{k'} + 3 \frac{k''}{k} \right) u_x^2 \\ &\quad + \frac{2}{3} (\lambda_1 k^4 + \lambda_2 k^2) + \frac{2}{3} u_x D_x^{-1} [2\lambda_1 k^3 k' + \lambda_2 k k']. \end{aligned} \quad (2.18)$$

Example 2.3: Let

$$\begin{aligned} m(u) &= \frac{3}{2} u^{-2}, \quad n(u) = -3u^{-1} \\ q(u) &= \lambda_1 u^2 + \lambda_2 u^{-2} + \lambda_3, \quad \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}, \end{aligned}$$

which satisfies system (2.4) – (2.5). Then (2.3) takes the form

$$\boxed{u_t = u_{xxx} - 3u^{-1}u_x u_{xx} + \frac{3}{2}u^{-2}u_x^3 + \lambda_1 u^2 u_x + \lambda_2 u^{-2}u_x + \lambda_3 u_x} \quad (2.19)$$

and by Proposition 2.1, eq. (2.19) admits the recursion operator

$$\begin{aligned} R[u] = & D_x^2 - 2u^{-1}u_x D_x - u^{-1}u_{xx} + u^{-2}u_x^2 + \frac{2\lambda_1}{3}u^2 + \frac{2\lambda_2}{3}u^{-2} \\ & + u_x D_x^{-1} \left[u^{-2}u_{xx} - u^{-3}u_x^2 + \frac{2\lambda_1}{3}u - \frac{2\lambda_2}{3}u^{-3} \right]. \end{aligned} \quad (2.20)$$

The next equation in the hierarchy is

$$\begin{aligned} u_t = & u_{xxxxx} - 5u^{-1}u_x u_{xxxx} - 10u^{-1}u_{xx} u_{xxx} + \frac{35}{2}u^{-2}u_x^2 u_{xxx} + \frac{55}{2}u^{-2}u_x u_{xx}^2 \\ & - \frac{95}{2}u^{-3}u_x^3 u_{xx} + \frac{135}{8}u^{-4}u_x^5 + \frac{5\lambda_1}{3} \left(u^2 u_{xxx} + u u_x u_{xx} - \frac{1}{2}u_x^3 \right) + \frac{5\lambda_2^2}{6}u^{-4}u_x \\ & + \frac{5\lambda_2}{3} \left(u^{-2}u_{xxx} - 7u^{-3}u_x u_{xx} + \frac{15}{2}u^{-4}u_x^3 \right) + \frac{5\lambda_1^2}{6}u^4 u_x + \frac{4\lambda_1 \lambda_2}{3}u_x \\ & + \lambda_3 \left(u_{xxx} - 3u^{-1}u_x u_{xx} + \frac{3}{2}u^{-2}u_x^3 + \lambda_1 u^2 u_x + \lambda_2 u^{-2}u_x \right). \end{aligned} \quad (2.21)$$

By the substitution

$$u \rightarrow k(u), \quad k \in \mathcal{C}^3,$$

eq. (2.19) takes the form

$$\boxed{u_t = u_{xxx} + 3 \left(\frac{k''}{k'} - \frac{k'}{k} \right) u_x u_{xx} + \left(\frac{k'''}{k'} - 3 \frac{k''}{k} + \frac{3}{2} \frac{(k')^2}{k^2} \right) u_x^3} \quad (2.22)$$

$$+ \lambda_1 k^2 u_x + \frac{\lambda_2}{k^2} u_x + \lambda_3 u_x$$

so that system (2.4) – (2.5) admits the solution

$$\begin{aligned} n(u) = & 3 \left(\frac{k''}{k'} - \frac{k'}{k} \right), \quad m(u) = \frac{k'''}{k'} - 3 \frac{k''}{k} + \frac{3}{2} \frac{(k')^2}{k^2} \\ q(u) = & \lambda_1 k^2 + \frac{\lambda_2}{k^2} + \lambda_3. \end{aligned}$$

By Proposition 2.1, eq. (2.22) admits the recursion operator

$$\begin{aligned} R[u] = & D_x^2 + 2 \left(\frac{k''}{k'} - \frac{k'}{k} \right) u_x D_x + \left(\frac{k'''}{k'} - \frac{k''}{k} \right) u_{xx} \\ & + \left(\frac{k'''}{k'} - 3 \frac{k''}{k} + \frac{(k')^2}{k^2} \right) u_x^2 + \frac{2}{3} \left(\lambda_1 k^2 + \frac{\lambda_2}{k^2} \right) \\ & + u_x D_x^{-1} \left[\frac{(k')^2}{k^2} u_{xx} + \left(\frac{k' k''}{k^2} - \frac{(k')^3}{k^3} \right) u_x^2 + \frac{2\lambda_1}{3} k k' - \frac{2\lambda_2}{3} \frac{k'}{k^3} \right]. \end{aligned} \quad (2.23)$$

Example 2.4: i) Let

$$n(u) = 0, \quad m(u) = -\frac{\beta^2}{8}$$

$$q(u) = \lambda_1 e^{\beta u} + \lambda_2 e^{-\beta u} + \lambda_3, \quad \beta, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R},$$

which satisfies system (2.4) – (2.5). Then (2.3) takes the form

$$\boxed{u_t = u_{xxx} - \frac{\beta^2}{8} u_x^3 + \lambda_1 e^{\beta u} u_x + \lambda_2 e^{-\beta u} u_x + \lambda_3 u_x} \quad (2.24)$$

and by Proposition 2.1, eq. (2.24) admits the recursion operator

$$\begin{aligned} R[u] = & D_x^2 - \frac{\beta^2}{4} u_x^2 + \frac{2\lambda_1}{3} e^{\beta u} + \frac{2\lambda_2}{3} e^{-\beta u} \\ & + u_x D_x^{-1} \left[\frac{\beta^2}{4} u_{xx} + \frac{\lambda_1 \beta}{3} e^{\beta u} - \frac{\lambda_2 \beta}{3} e^{-\beta u} \right]. \end{aligned} \quad (2.25)$$

This result is also given in [7]. The next equation in the hierarchy is

$$\begin{aligned} u_t = & u_{xxxxx} - \frac{5\beta^2}{8} u_x^2 u_{xxx} - \frac{5\beta^2}{8} u_x u_{xx}^2 + \frac{3\beta^4}{128} u_x^5 + \frac{5\lambda_1^2}{6} e^{2\beta u} u_x + \frac{4\lambda_1 \lambda_2}{3} u_x \\ & + \frac{5\lambda_1}{3} e^{\beta u} \left(u_{xxx} + 2\beta u_x u_{xx} + \frac{3\beta^2}{8} u_x^3 \right) + \frac{5\lambda_2}{3} e^{-\beta u} \left(u_{xxx} - 2\beta u_x u_{xx} + \frac{3\beta^2}{8} u_x^3 \right) \\ & + \frac{5\lambda_2^2}{6} e^{-2\beta u} u_x + \lambda_3 \left(u_{xxx} - \frac{\beta^2}{8} u_x^3 + \lambda_1 e^{\beta u} u_x + \lambda_2 e^{-\beta u} u_x \right). \end{aligned} \quad (2.26)$$

By the substitution

$$u \rightarrow k(u), \quad k \in \mathcal{C}^3,$$

eq. (2.24) takes the form

$$\boxed{u_t = u_{xxx} + 3 \frac{k''}{k'} u_x u_{xx} + \left(\frac{k'''}{k'} - \frac{\beta^2}{8} (k')^2 \right) u_x^3 + \lambda_1 e^{\beta k} u_x + \lambda_2 e^{-\beta k} u_x + \lambda_3 u_x} \quad (2.27)$$

so that system (2.4) – (2.5) admits the solution

$$n(u) = 3 \frac{k''}{k'}, \quad m(u) = \frac{k'''}{k'} - \frac{\beta^2}{8} (k')^2$$

$$q(u) = \lambda_1 e^{\beta k} + \lambda_2 e^{-\beta k} + \lambda_3.$$

By Proposition 2.1, eq. (2.27) admits the recursion operator

$$\begin{aligned} R[u] = & D_x^2 + 2\frac{k''}{k'}u_x D_x + \frac{k''}{k'}u_{xx} + \left(\frac{k'''}{k'} - \frac{\beta^2}{4}(k')^2\right)u_x^2 + \frac{2\lambda_1}{3}e^{\beta k} + \frac{2\lambda_2}{3}e^{-\beta k} \\ & + u_x D_x^{-1} \left[\frac{\beta^2}{4}((k')^2 u_{xx} + k'k''u_x^2) + \frac{\beta\lambda_1}{3}k'e^{\beta k} - \frac{\beta\lambda_2}{3}k'e^{-\beta k} \right]. \end{aligned} \quad (2.28)$$

Example 2.4: ii) Closely related to Example 2.4 i) is the solution

$$n(u) = 0, \quad m(u) = \frac{\beta^2}{8}$$

$$q(u) = \lambda_1 \cos(\beta u) + \lambda_2 \sin(\beta u) + \lambda_3, \quad \beta, \lambda_1, \lambda_2, \lambda_3 \in \mathfrak{R},$$

which satisfies the system (2.4) – (2.5). Then (2.3) takes the form

$$\boxed{u_t = u_{xxx} + \frac{\beta^2}{8}u_x^3 + \lambda_1 \cos(\beta u)u_x + \lambda_2 \sin(\beta u)u_x + \lambda_3 u_x} \quad (2.29)$$

and by Proposition 2.1, eq. (2.29) admits the recursion operator

$$\begin{aligned} R[u] = & D_x^2 + \frac{\beta^2}{4}u_x^2 + \frac{2}{3}\lambda_1 \cos(\beta u) + \frac{2}{3}\lambda_2 \sin(\beta u) \\ & - u_x D_x^{-1} \left[\frac{\beta^2}{4}u_{xx} + \frac{\lambda_1\beta}{3}\sin(\beta u) - \frac{\lambda_2\beta}{3}\cos(\beta u) \right]. \end{aligned} \quad (2.30)$$

The next equation in this hierarchy is

$$\begin{aligned} u_t = & u_{xxxxx} + \frac{5\beta^2}{8}u_x^2 u_{xxx} + \frac{5\beta^2}{8}u_x u_{xx}^2 + \frac{3\beta^4}{128}u_x^5 + \frac{5\lambda_2^2}{6}\sin^2(\beta u)u_x \\ & + \frac{5\lambda_1\lambda_2}{3}\cos(\beta u)\sin(\beta u)u_x + \frac{5\lambda_1^2}{6}\cos^2(\beta u)u_x - \frac{\lambda_1^2}{6}u_x - \frac{\lambda_2^2}{6}u_x \\ & + \frac{5\lambda_1}{3} \left(\cos(\beta u)u_{xxx} - 2\beta \sin(\beta u)u_x u_{xx} - \frac{3\beta^2}{8}\cos(\beta u)u_x^3 \right) \\ & + \frac{5\lambda_2}{3} \left(\sin(\beta u)u_{xxx} + 2\beta \cos(\beta u)u_x u_{xx} - \frac{3\beta^2}{8}\sin(\beta u)u_x^3 \right) \\ & + \lambda_3 \left(u_{xxx} + \frac{\beta^2}{8}u_x^3 + \lambda_1 \cos(\beta u)u_x + \lambda_2 \sin(\beta u)u_x \right). \end{aligned} \quad (2.31)$$

By the substitution

$$u \rightarrow k(u), \quad k \in C^3,$$

eq. (2.29) takes the form

$$\boxed{u_t = u_{xxx} + 3\frac{k''}{k'}u_x u_{xx} + \left(\frac{k'''}{k'} + \frac{\beta^2}{8}(k')^2\right)u_x^3 + \lambda_1 \cos(\beta k)u_x + \lambda_2 \sin(\beta k)u_x + \lambda_3 u_x} \quad (2.32)$$

so that system (2.4) – (2.5) admits the solution

$$n(u) = 3\frac{k''}{k'}, \quad m(u) = \frac{k'''}{k'} + \frac{\beta^2}{8}(k')^2$$

$$q(u) = \lambda_1 \cos(\beta k) + \lambda_2 \sin(\beta k) + \lambda_3.$$

By Proposition 2.1, eq. (2.32) admits the recursion operator

$$\begin{aligned} R[u] = & D_x^2 + 2\frac{k''}{k'}u_x D_x + \frac{k''}{k'}u_{xx} + \left(\frac{k'''}{k'} + \frac{\beta^2}{4}(k')^2\right)u_x^2 \\ & + \frac{2\lambda_1}{3}\cos(\beta k) + \frac{2\lambda_2}{3}\sin(\beta k) \\ & - u_x D_x^{-1} \left[\frac{\beta^2}{4}((k')^2 u_{xx} + k'k''u_x^2) + \frac{\beta\lambda_1}{3}k' \sin(\beta k) - \frac{\beta\lambda_2}{3}k' \cos(\beta k) \right]. \end{aligned} \quad (2.33)$$

Our last example is on the well known KdV and modified KdV equation, which should obviously be contained in Proposition 2.1. The recursion operators for these equations are well known (see for example [7])

Example 2.5: Let

$$m(u) = 0, \quad n(u) = 0$$

$$q(u) = \lambda_1 u^2 + \lambda_2 u + \lambda_3, \quad \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R},$$

which satisfies system (2.4) – (2.5). Then (2.3) takes the form

$$\boxed{u_t = u_{xxx} + \lambda_1 u^2 u_x + \lambda_2 u u_x + \lambda_3 u_x} \quad (2.34)$$

and admits the recursion operator

$$R[u] = D_x^2 + \frac{2\lambda_1}{3}u^2 + \frac{2\lambda_2}{3}u + \frac{1}{3}u_x D_x^{-1} [2\lambda_1 u + \lambda_2]. \quad (2.35)$$

With $\lambda_1 = 0$, eq. (2.34) is the KdV equation and for $\lambda_2 = 0$ eq. (2.34) is known as the modified KdV equation. This result is also given in [7]. The next equation in this hierarchy is

$$\begin{aligned} u_t = & u_{xxxxx} + \frac{5\lambda_1}{3}(u^2 u_{xxx} + 4u u_x u_{xx} + u_x^3) + \frac{5\lambda_2}{3}(u u_{xxx} + 2u_x u_{xx}) \\ & + \frac{5\lambda_1^2}{6}u^4 u_x + \frac{5\lambda_1 \lambda_2}{3}u^3 u_x + \frac{5\lambda_2^2}{6}u^2 u_x + \lambda_3(u_{xxx} + \lambda_1 u^2 u_x + \lambda_2 u u_x). \end{aligned} \quad (2.36)$$

By the substitution

$$u \rightarrow k(u), \quad k \in \mathcal{C}^3$$

eq. (2.34) takes the form

$$\boxed{u_t = u_{xxx} + 3\frac{k''}{k'}u_x u_{xx} + \frac{k'''}{k'}u_x^3 + \lambda_1 k^2 u_x + \lambda_2 k u_x + \lambda_3 u_x} \quad (2.37)$$

so that system (2.4) – (2.5) admits the solution

$$\begin{aligned} m(u) &= \frac{k'''}{k'}, & n(u) &= 3\frac{k''}{k'} \\ q(u) &= \lambda_1 k^2 + \lambda_2 k + \lambda_3, & \lambda_1, \lambda_2, \lambda_3 &\in \mathfrak{R}. \end{aligned}$$

By Proposition 2.1, eq. (2.37) admits the recursion operator

$$\begin{aligned} R[u] &= D_x^2 + 2\frac{k''}{k'}u_x D_x + \frac{k''}{k'}u_{xx} + \frac{k'''}{k'}u_x^2 + \frac{2}{3}(\lambda_1 k^2 + \lambda_2 k) \\ &\quad + \frac{1}{3}u_x D_x^{-1} [2\lambda_1 k k' + \lambda_2 k']. \end{aligned} \quad (2.38)$$

3 Recursion operators for the quasilinear equation (1.2)

The recursion operator Ansatz (1.3) for (1.2) leads to

Proposition 3.1 *Evolution equations of the form*

$$u_t = u^3 u_{xxx} + n(u)u_x u_{xx} + m(u)u_x^3 + r(u)u_{xx} + p(u)u_x^2 + q(u)u_x + s(u),$$

which admit recursion operators of the form (1.3), viz.

$$R[u] = G D_x^2 + Q D_x + H + \sum_j I_j D_x^{-1} J_j$$

and which are not linearisable by a x -generalised hodograph transformation, are exhausted by the following two cases:

I) *The equation*

$$\boxed{u_t = u^3 u_{xxx} + 3u^2 u_x u_{xx} + \lambda_1 u^3 u_x + \lambda_2 u^2 u_x + \lambda_3 u_x} \quad \lambda_j \in \mathfrak{R} \quad (3.1)$$

admits the recursion operator

$$\begin{aligned} R[u] &= u^2 D_x^2 + u u_x D_x + 2u u_{xx} + \lambda_1 u^2 + \frac{4\lambda_2}{3}u \\ &\quad + (u^3 u_{xxx} + 3u^2 u_x u_{xx} + \lambda_1 u^3 u_x + \lambda_2 u^2 u_x) D_x^{-1} u^{-2} - \frac{\lambda_2}{3}u_x D_x^{-1}. \end{aligned} \quad (3.2)$$

II) *The equation*

$$\boxed{u_t = u^3 u_{xxx} + \lambda_1 u^3 u_x + \lambda_2 u^{-1} u_x + \lambda_3 u_x} \quad \lambda_j \in \mathbb{R} \quad (3.3)$$

admits the recursion operator

$$\begin{aligned} R[u] &= u^2 D_x^2 - u u_x D_x + u u_{xx} + \lambda_1 u^2 + \frac{\lambda_2}{3} u^{-2} \\ &+ \left(u^3 u_{xxx} + \lambda_1 u^3 u_x + \lambda_2 u^{-1} u_x \right) D_x^{-1} u^{-2} - \frac{4\lambda_2}{3} u_x D_x^{-1} u^{-3}. \end{aligned} \quad (3.4)$$

To prove Proposition 3.1 one needs to verify commutator relation (1.4) for eq. (1.2) and the recursion operator Ansatz (1.3). This is done by direct calculations, so we do not show the details here.

We apply the recursion operators of Proposition 3.1 on the t -translation symmetry, i.e. $R[u]u_t$, to obtain the next equation in the hierarchies. For eq. (3.1) the next equation in the hierarchy is

$$\begin{aligned} u_t &= u^5 u_{xxxxx} + 10u^4 u_x u_{xxxx} + 15u^4 u_{xx} u_{xxx} + 25u^3 u_x^2 u_{xxx} + 30u^3 u_x u_{xx}^2 \\ &+ 15u^2 u_x^3 u_{xx} + \frac{5\lambda_1}{2} (u^5 u_{xxx} + 7u^4 u_x u_{xx} + 4u^3 u_x^3) \\ &+ \frac{10\lambda_2}{3} \left(u^4 u_{xxx} + 5u^3 u_x u_{xx} + \frac{3}{2} u^2 u_x^3 \right) \\ &+ \frac{3\lambda_1^2}{2} u^5 u_x + \frac{15\lambda_1 \lambda_2}{4} u^4 u_x + \frac{20\lambda_2^2}{9} u^3 u_x \end{aligned} \quad (3.5)$$

and for eq. (3.3) the next equation in the hierarchy is

$$\begin{aligned} u_t &= u^5 u_{xxxxx} + 5u^4 u_x u_{xxxx} + 5u^4 u_{xx} u_{xxx} + \frac{5}{2} u^3 u_x^2 u_{xxx} \\ &+ \frac{5\lambda_1}{2} (u^5 u_{xxx} + 4u^4 u_x u_{xx} + u^3 u_x^3) + \frac{5\lambda_2}{6} (u u_{xxx} - 4u_x u_{xx} + 3u^{-1} u_x^3) \\ &+ \frac{3\lambda_1^2}{2} u^5 u_x + \frac{5\lambda_2^2}{18} u^{-3} u_x. \end{aligned} \quad (3.6)$$

We now use the substitution

$$u \rightarrow k(u)$$

for the two equations of Proposition 3.1. For eq. (3.1) we obtain

$$\boxed{u_t = k^3 u_{xxx} + 3 \left(k^3 \frac{k''}{k'} + k^2 k' \right) u_x u_{xx} + \left(k^3 \frac{k'''}{k'} + 3k^2 k'' \right) u_x^3 + \lambda_1 k^3 u_x + \lambda_2 k^2 u_x + \lambda_3 u_x} \quad (3.7)$$

with the recursion operator

$$\begin{aligned}
R[u] = & k^2 D_x^2 + \left(2k^2 \frac{k''}{k'} + kk'\right) u_x D_x + \left(k^2 \frac{k''}{k'} + 2kk'\right) u_{xx} + \lambda_1 k^2 + \frac{4\lambda_2}{3} k \\
& + \left(k^2 \frac{k'''}{k'} + 3kk''\right) u_x^2 + \left\{k^3 u_{xxx} + 3\left(k^3 \frac{k''}{k'} + k^2 k'\right) u_x u_{xx} \right. \\
& \left. + \left(k^3 \frac{k'''}{k'} + 3k^2 k''\right) u_x^3 + \lambda_1 k^3 u_x + \lambda_2 k^2 u_x\right\} D_x^{-1} \frac{k'}{k^2} - \frac{\lambda_2}{3} u_x D_x^{-1} k'. \quad (3.8)
\end{aligned}$$

For eq. (3.3) we obtain

$$\boxed{u_t = k^3 u_{xxx} + 3k^3 \frac{k''}{k'} u_x u_{xx} + k^3 \frac{k'''}{k'} u_x^3 + \lambda_1 k^3 u_x + \frac{\lambda_2}{k} u_x + \lambda_3 u_x} \quad (3.9)$$

and its recursion operator is

$$\begin{aligned}
R[u] = & k^2 D_x^2 + \left(2k^2 \frac{k''}{k'} - kk'\right) u_x D_x + \left(k^2 \frac{k''}{k'} + kk'\right) u_{xx} + \lambda_1 k^2 \\
& + k^2 \frac{k'''}{k'} u_x^2 + \frac{\lambda_2}{3} \frac{1}{k^2} - \frac{4\lambda_2}{3} u_x D_x^{-1} \frac{k'}{k^3} \\
& + \left(k^3 u_{xxx} + 3k^3 \frac{k''}{k'} u_x u_{xx} + k^3 \frac{k'''}{k'} u_x^3 + \lambda_1 k^3 u_x + \frac{\lambda_2}{k} u_x\right) D_x^{-1} \frac{k'}{k^2}. \quad (3.10)
\end{aligned}$$

4 Transformations between equations

4.1 From semilinear to quasilinear equations

We make use of the x -generalised hodograph transformation (1.6) to transform those semilinear integrable equations, given by Proposition 2.1, to quasilinear equations within the class of autonomous evolution equations

$$u_t = F(u, u_x, u_{xx}, u_{xxx}).$$

Equation (2.7) given by Proposition 2.1 Case I, namely

$$u_t = u_{xxx} + 3 \frac{k''}{k'} u_x u_{xx} + \left(\frac{k'''}{k'} + \lambda_1 (k')^2\right) u_x^3 + \lambda_2 k' u_x^2 + \lambda_3 u_x + \frac{\lambda_4}{k'},$$

is transformed by the x -generalised hodograph transformation

$${}_3\mathbf{H} : \begin{cases} dx(\tilde{x}, \tilde{t}) = \tilde{k}^{-1} d\tilde{x} \\ \quad + \left\{ -\tilde{k}\tilde{k}' \tilde{u}_{\tilde{x}\tilde{x}} - [(\tilde{k}')^2 + \tilde{k}\tilde{k}''] \tilde{u}_{\tilde{x}}^2 - \lambda_1 \tilde{k}^2 - \lambda_2 \tilde{k} - \lambda_4 \tilde{k}^{-1} - \lambda_3 \right\} d\tilde{t} \\ dt(\tilde{x}, \tilde{t}) = d\tilde{t} \\ k(u) = \tilde{x}, \end{cases} \quad (4.1)$$

to the quasilinear equation (3.7), namely

$$\begin{aligned} \tilde{u}_{\tilde{t}} = & \tilde{k}^3 \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + 3 \left(\tilde{k}^3 \frac{\tilde{k}''}{\tilde{k}'} + \tilde{k}^2 \tilde{k}' \right) \tilde{u}_{\tilde{x}} \tilde{u}_{\tilde{x}\tilde{x}} + \left(\tilde{k}^3 \frac{\tilde{k}'''}{\tilde{k}'} + 3\tilde{k}^2 \tilde{k}'' \right) \tilde{u}_{\tilde{x}}^3 \\ & + 2\lambda_1 \tilde{k}^3 \tilde{u}_{\tilde{x}} + \lambda_2 \tilde{k}^2 \tilde{u}_{\tilde{x}} - \lambda_4 \tilde{u}_{\tilde{x}}. \end{aligned}$$

We recall that eq. (3.7) admits the recursion operator (3.8).

The only integrable equation of Proposition 2.1, Case II, that can be transformed by a x -generalised hodograph transformation within the class of autonomous equations, is the KdV equation (2.37) with $\lambda_1 = 0$, i.e.

$$u_t = u_{xxx} + 3 \frac{k''}{k'} u_x u_{xx} + \frac{k'''}{k'} u_x^3 + \lambda_2 k u_x + \lambda_3 u_x. \quad (4.2)$$

By the the x -generalised hodograph transformation

$${}_3\mathbf{H} : \begin{cases} dx(\tilde{x}, \tilde{t}) = \tilde{k}^{-1} d\tilde{x} \\ \quad + \left\{ -\tilde{k}\tilde{k}'\tilde{u}_{\tilde{x}\tilde{x}} - \left((\tilde{k}')^2 + \tilde{k}\tilde{k}'' \right) \tilde{u}_{\tilde{x}}^2 - \lambda_2 \tilde{x} - \lambda_3 \right\} d\tilde{t} \\ dt(\tilde{x}, \tilde{t}) = d\tilde{t} \\ k(u) = \tilde{x}, \end{cases} \quad (4.3)$$

we obtain for (4.2) the following quasilinear equation

$$\boxed{\tilde{u}_{\tilde{t}} = \tilde{k}^3 \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + 3 \left(\tilde{k}^3 \frac{\tilde{k}''}{\tilde{k}'} + \tilde{k}^2 \tilde{k}' \right) \tilde{u}_{\tilde{x}} \tilde{u}_{\tilde{x}\tilde{x}} + \left(\tilde{k}^3 \frac{\tilde{k}'''}{\tilde{k}'} + 3\tilde{k}^2 \tilde{k}'' \right) \tilde{u}_{\tilde{x}}^3 + \lambda_2 \frac{\tilde{k}^2}{\tilde{k}'}} \quad (4.4)$$

Equation (4.4) does not admit a recursion operator of the form (1.3).

It is interesting to note that both (3.3) with $\lambda_1 = \lambda_2 = \lambda_3 = 0$, i.e. the well known *Harry-Dym equation*

$$u_t = u^3 u_{xxx}, \quad (4.5)$$

and (2.1) with $\lambda_2 = \lambda_4 = 0$, i.e the potential modified KdV equation

$$\tilde{u}_{\tilde{t}} = \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + \lambda_1 \tilde{u}_{\tilde{x}}^3 + \lambda_3 \tilde{u}_{\tilde{x}}, \quad (4.6)$$

can be transformed to eq. (4.4) by a x -generalised hodograph transformation. The composition of those two x -generalised hodograph transformations results in the following transformation between eq. (4.5) and eq. (4.6):

$${}_3\mathbf{H} : \begin{cases} dx(\tilde{x}, \tilde{t}) = \frac{1}{\alpha} e^{\alpha \tilde{u}} d\tilde{x} + \frac{1}{\alpha} e^{\alpha \tilde{u}} \{ \alpha \tilde{u}_{\tilde{x}\tilde{x}} + \lambda_1 \tilde{u}_{\tilde{x}}^2 + \lambda_3 \} d\tilde{t} \\ dt(\tilde{x}, \tilde{t}) = d\tilde{t} \\ u(x, t) = \frac{1}{\alpha} e^{\alpha \tilde{u}}, \quad \alpha^2 = -2\lambda_1, \quad \alpha = \text{nonzero constant.} \end{cases} \quad (4.7)$$

We remark that this relation between the Harry-Dym and the modified KdV was also given in [10].

4.2 Potential forms for semilinear equations of Proposition 2.1

We now consider eq.(2.14) of Example 2.2 and eq. (2.24) of Example 2.4 i). Our aim is to perform the following change of coordinates successively:

$$(x, t, u(x, t)) \mapsto (x, t, v(x, t)) \quad \text{Potential Form (PF) with } v_x = h(u) \quad (4.8)$$

$$(x, t, v(x, t)) \mapsto (X, t, V(X, t)) \quad \text{Hodograph Transformation (HT)} \quad (4.9)$$

$$(X, t, V(X, t)) \mapsto (X, t, W(X, t)) \quad \text{with } W = V_X \quad (4.10)$$

$$(X, t, W(X, t)) \mapsto (\tilde{x}, \tilde{t}, \tilde{u}(\tilde{x}, \tilde{t})) \quad x\text{-Generalised Hodograph Transformation} \quad (4.11)$$

The hodograph transformation (4.9) has the following explicit form:

$$X = v(x, t), \quad T = t, \quad V(X, t) = x. \quad (4.12)$$

4.2.1 We consider eq. (2.14), i.e.

$$u_t = u_{xxx} + 3u^{-1}u_x u_{xx} + \lambda_1 u^4 u_x + \lambda_2 u^2 u_x + \lambda_3 u_x,$$

which admits two potential forms. Those are discussed separately under 4.2.1 i) and 4.2.1 ii) below:

4.2.1 i) For eq. (2.14) we consider the change of coordinates (4.8) and set

$$v_x = u^4$$

The potential equation is

$$\boxed{v_t = v_{xxx} - \frac{3}{4}v_x^{-1}v_{xx}^2 + \frac{1}{2}\lambda_1 v_x^2 + \frac{2}{3}\lambda_2 v_x^{3/2} + \lambda_3 v_x + C} \quad C \in \mathbb{R}. \quad (4.13)$$

Under the hodograph transformation (4.12), eq. (4.13) takes the form

$$V_t = V_X^{-3}V_{XXX} - \frac{9}{4}V_X^{-4}V_{XX}^2 - \frac{1}{2}\lambda_1 V_X^{-1} - \frac{2}{3}\lambda_2 V_X^{-1/2} - C V_X - \lambda_3. \quad (4.14)$$

We remark that if we now apply the x -generalised hodograph transformation (1.6) on (4.14) then we obtain (2.14), since

$$(x\text{-Generalised Hodograph Transformation})^{-1} = \text{HT} \circ \text{PF}. \quad (4.15)$$

We now apply (4.10), which transforms eq. (4.14) to

$$\begin{aligned} W_t = & W^{-3}W_{XXX} - \frac{15}{2}W^{-4}W_X W_{XX} + 9W^{-5}W_X^3 + \frac{1}{2}\lambda_1 W^{-2}W_X \\ & + \frac{1}{3}\lambda_2 W^{-3/2}W_X - C W_X. \end{aligned} \quad (4.16)$$

It is worth to note that, with $\lambda_1 = 0$, eq. (4.16) transforms to (4.4) by the x -generalised hodograph transformation

$${}_{\mathbf{3H}} : \begin{cases} dX(\tilde{x}, \tilde{t}) = \tilde{x}^2 \tilde{k}^{-1} d\tilde{x} \\ \quad + \left\{ -\tilde{k}^2 + 2\tilde{x} \tilde{k} \tilde{k}' \tilde{u}_{\tilde{x}} - \tilde{x}^2 [(\tilde{k}')^2 + \tilde{k} \tilde{k}''] \tilde{u}_{\tilde{x}}^2 - \tilde{x}^2 \tilde{k} \tilde{k}' \tilde{u}_{\tilde{x}\tilde{x}} + \frac{\lambda_2}{3} \tilde{x}^3 + C \right\} d\tilde{t} \\ dt(\tilde{x}, \tilde{t}) = d\tilde{t} \\ W(X, t) = \tilde{x}^{-2}. \end{cases} \quad (4.17)$$

We recall that (4.4) can be transformed to the KdV equation (see (4.2) and (4.4)).

4.2.1 ii) For the second potential form of (2.14) we set

$$v_x = u^2,$$

which leads to

$$v_t = v_{xxx} + \frac{1}{3} \lambda_1 v_x^3 + \frac{1}{2} \lambda_2 v_x^2 + \lambda_3 v_x + C. \quad (4.18)$$

Note that by a simple change of constants eq.(4.18) is equivalent to eq. (2.1) of Proposition 2.1. Under the hodograph transformation (4.12), eq. (4.18) takes the form

$$V_t = V_X^{-3} V_{XXX} - 3V_X^{-4} V_{XX}^2 - \frac{1}{3} \lambda_1 V_X^{-2} - \frac{1}{2} \lambda_2 V_X^{-1} - C V_X - \lambda_3. \quad (4.19)$$

We now apply (4.10), so eq. (4.19) becomes

$$\begin{aligned} W_t &= W^{-3} W_{XXX} - 9W^{-4} W_X W_{XX} + 12W^{-5} W_X^3 + \frac{2}{3} \lambda_1 W^{-3} W_X \\ &\quad + \frac{1}{2} \lambda_2 W^{-2} W_X - C W_X \end{aligned} \quad (4.20)$$

Equation (4.20) is equivalent to (3.1) with $C = \lambda_3$ and

$$u(x, t) = W^{-1}(X, t), \quad X = x.$$

4.2.2 We now consider eq. (2.24) of Example 2.4 i, i.e.

$$u_t = u_{xxx} - \frac{\beta^2}{8} u_x^3 + \lambda_1 e^{\beta u} u_x + \lambda_2 e^{-\beta u} u_x + \lambda_3 u_x.$$

For the change of coordinates (4.8) we set

$$v_x = \exp \left\{ \frac{\beta u}{2} \right\}$$

and obtain the potential equation

$$\boxed{v_t = v_{xxx} - \frac{3}{2} v_x^{-1} v_{xx}^2 + \frac{1}{3} \lambda_1 v_x^3 - \lambda_2 v_x^{-1} + \lambda_3 v_x + C} \quad C \in \mathbb{R}. \quad (4.21)$$

We remark that, with

$$\lambda_1 = \lambda_2 = \lambda_3 = C = 0,$$

eq. (4.21) is known as the *Kirchever-Novikov equation* [4] and admits the recursion operator [12]

$$\begin{aligned} R[v] = & D_x^2 - 2v_x^{-1}v_{xx}D_x + (v_x^{-1}v_{xxx} - v_x^{-2}v_{xx}^2) \\ & - v_x D_x^{-1} [3v_x^{-4}v_{xx}^3 - 4v_x^{-3}v_{xx}v_{xxx} + v_x^{-2}v_{xxxx}]. \end{aligned} \quad (4.22)$$

Under the hodograph transformation (4.12), eq. (4.21) takes the form

$$V_t = V_X^{-3}V_{XXX} - \frac{3}{2}V_X^{-4}V_{XX}^2 - \frac{1}{3}\lambda_1 V_X^{-2} + \lambda_2 V_X^2 - CV_X - \lambda_3. \quad (4.23)$$

We now apply (4.10), which transforms eq. (4.23) to

$$\begin{aligned} W_t = & W^{-3}W_{XXX} - 6W^{-4}W_XW_{XX} + 6W^{-5}W_X^3 + \frac{2}{3}\lambda_1 W^{-3}W_X \\ & + \lambda_2 WW_X - CW_X. \end{aligned} \quad (4.24)$$

Equation (4.24) is equivalent to (3.3) with $C = \lambda_3$ and

$$u(x, t) = W^{-1}(X, t), \quad X = x.$$

4.3 Potential forms for quasilinear equations of Proposition 3.1

We now consider the quasilinear equations of Proposition 2, i.e. equations (3.1) and (3.3). As in the previous subsection we apply the sequence of coordinate transformations (4.8)–(4.11).

4.3.1 Consider eq. (3.1), i.e.

$$u_t = u^3u_{xxx} + 3u^2u_xu_{xx} + \lambda_1u^3u_x + \lambda_2u^2u_x + \lambda_3u_x \quad \lambda_j \in \mathfrak{R}$$

which admits two potential forms. Those are discussed separately under 4.3.1 i) and 4.3.1 ii) below:

4.3.1 i) For eq. (3.1) we consider the change of coordinates (4.8) and set

$$v_x = u^{-1}$$

The potential equation is

$$\boxed{v_t = v_x^{-3}v_{xxx} - 3v_x^{-4}v_{xx}^2 - \frac{\lambda_1}{2}v_x^{-2} - \lambda_2v_x^{-1} + \lambda_3v_x + C} \quad C \in \mathfrak{R}. \quad (4.25)$$

Under the hodograph transformation (4.12), eq. (4.25) takes the form

$$V_t = V_{XXX} + \frac{\lambda_1}{2}V_X^3 + \lambda_2V_X^2 - CV_X - \lambda_3 \quad (4.26)$$

which is just eq. (2.1).

4.3.1 ii) To obtain the second potential form for (3.1) we set

$$v_x = u.$$

The potential equation is

$$\boxed{v_t = v_x^3 v_{xxx} + \frac{\lambda_1}{4} v_x^4 + \frac{\lambda_2}{3} v_x^3 + \lambda_3 v_x + C} \quad C \in \mathfrak{R}. \quad (4.27)$$

Under the hodograph transformation (4.12), eq. (4.27) takes the form

$$V_t = V_X^{-6} V_{XXX} - 3V_X^{-7} V_{XX}^2 - \frac{\lambda_1}{4} V_X^{-3} - \frac{\lambda_2}{3} V_X^{-2} - C V_X - \lambda_3. \quad (4.28)$$

We now apply (4.10), which transforms eq. (4.28) to

$$\begin{aligned} W_t = & W^{-6} W_{XXX} - 12W^{-7} W_X W_{XX} + 21W^{-8} W_X^3 \\ & + \frac{3\lambda_1}{4} W^{-4} W_X + \frac{2\lambda_2}{3} W^{-3} W_X - C W_X. \end{aligned} \quad (4.29)$$

Note that, with $\lambda_1 = 0$, eq. (4.29) can be transformed to (4.4) by the x -generalised hodograph transformation

$${}_3\mathbf{H} : \begin{cases} dX(\tilde{x}, \tilde{t}) = \frac{1}{4} \tilde{x}^2 \tilde{k}^{-1} d\tilde{x} \\ \quad + \left\{ -\frac{1}{4} \tilde{k}^2 + \frac{1}{2} \tilde{x} \tilde{k} \tilde{k}' \tilde{u}_{\tilde{x}} - \frac{\tilde{x}^2}{4} [(\tilde{k}')^2 + \tilde{k} \tilde{k}''] \tilde{u}_{\tilde{x}}^2 - \frac{\tilde{x}^2}{4} \tilde{k} \tilde{k}' \tilde{u}_{\tilde{x}\tilde{x}} - \frac{\lambda_2}{12} \tilde{x}^3 + C \right\} d\tilde{t} \\ dt(\tilde{x}, \tilde{t}) = d\tilde{t} \\ W(X, t) = 2\tilde{x}^{-1}. \end{cases}$$

We recall that (4.4) can be transformed to the KdV equation (4.2).

4.3.2 We now consider (3.3), i.e.

$$u_t = u^3 u_{xxx} + \lambda_1 u^3 u_x + \lambda_2 u^{-1} u_x + \lambda_3 u_x, \quad \lambda_j \in \mathfrak{R}$$

which admits two potential forms. Those are discussed separately under 4.3.2 i) and 4.3.2 ii) below:

4.3.2 i) For eq. (3.3) we consider the change of coordinates (4.8) and set

$$v_x = u^{-1}.$$

The potential equation is

$$\boxed{v_t = v_x^{-3} v_{xxx} - \frac{3}{2} v_x^{-4} v_{xx}^2 - \frac{\lambda_1}{2} v_x^{-2} + \frac{\lambda_2}{2} v_x^2 + \lambda_3 v_x + C} \quad C \in \mathfrak{R}. \quad (4.30)$$

Under the hodograph transformation (4.12), eq. (4.30) takes the form

$$V_t = V_{XXX} - \frac{3}{2}V_X^{-1}V_{XX}^2 + \frac{\lambda_1}{2}V_X^3 - \frac{\lambda_2}{2}V_X^{-1} - CV_X - \lambda_3. \quad (4.31)$$

We now apply (4.10), by which eq. (4.31) becomes

$$\begin{aligned} W_t = W_{XXX} - 3W^{-1}W_XW_{XX} + \frac{3}{2}W^{-2}W_X^3 + \frac{3}{2}\lambda_1W^2W_X \\ + \frac{1}{2}\lambda_2W^{-2}W_X - CW_X. \end{aligned} \quad (4.32)$$

Equation (4.32) is just eq.(2.19) of Example 2.3.

4.3.2 ii) For the second potential form of eq. (3.3) we set

$$v_x = u^{-2}.$$

The potential equation is

$$\boxed{v_t = v_x^{-3/2}v_{xxx} - \frac{3}{2}v_x^{-5/2}v_{xx}^2 - 2\lambda_1v_x^{-1/2} + \frac{2\lambda_2}{3}v_x^{3/2} + \lambda_3v_x + C} \quad C \in \mathfrak{R}. \quad (4.33)$$

Equation (4.33) is a slight generalisation of the *Cavalcante-Tenenblat equation* [3] and admits the recursion operator

$$\begin{aligned} R[v] = v_x^{-1}D_x^2 - \frac{3}{2}v_x^{-2}v_{xx}D_x - \frac{1}{2}v_x^{-2}v_{xxx} + \frac{3}{4}v_x^{-3}v_{xx}^2 + \lambda_1v_x^{-1} + \frac{\lambda_2}{3}v_x \\ - \frac{1}{4}\left(v_x^{-3/2}v_{xxx} - \frac{3}{2}v_x^{-5/2}v_{xx}^2 - 2\lambda_1v_x^{-1/2} + \frac{2\lambda_2}{3}\right)D_x^{-1}v_x^{-3/2}v_{xx}. \end{aligned} \quad (4.34)$$

Under the hodograph transformation (4.12), eq. (4.33) takes the form

$$V_t = V_X^{-3/2}V_{XXX} - \frac{3}{2}V_X^{-5/2}V_{XX}^2 - \frac{2\lambda_2}{3}V_X^{-1/2} + 2\lambda_1V_X^{3/2} - CV_X - \lambda_3. \quad (4.35)$$

We note that with

$$\lambda_2 = 3\lambda_1, \quad C = -\lambda_3$$

eq. (4.33) is invariant under the hodograph transformation (4.12). We now apply (4.10) to (4.35), so that eq. (4.35) becomes

$$\begin{aligned} W_t = W^{-3/2}W_{XXX} - \frac{9}{2}W^{-5/2}W_XW_{XX} + \frac{15}{4}W^{-7/2}W_X^3 + \lambda_1W^{-3/2}W_X \\ + \lambda_2W^{1/2}W_X + \lambda_3W_X. \end{aligned} \quad (4.36)$$

Equation (4.36) is equivalent to eq. (3.3) of Proposition 3.1 with

$$u(x, t) = W^{-1/2}(X, t), \quad X = x.$$

4.4 A nonlocal recursion operator

The following equation was introduced in [2, 9]:

$$\boxed{u_t = u_{xxx} + 3u^2 u_{xx} + 9u u_x^2 + 3u^4 u_x} \quad (4.37)$$

which is a special case of our semilinear equation (1.1). As pointed out in [1, 11], the linearisation of eq. (4.37) can be achieved by a nonlocal transformation. We establish this linearisation using a combination of potential forms and x -generalised hodograph transformations. We then use these transformations to calculate the recursion operator for (4.37). The recursion operator for (4.37) turns out to be nonlocal, which explains why this result is not part of Proposition 2.1.

First we write (4.37) in potential form and set

$$v_x = u^2.$$

This leads to

$$\boxed{v_t = v_{xxx} - \frac{3}{4}v_x^{-1}v_{xx}^2 + 3v_x v_{xx} + v_x^3} \quad (4.38)$$

Now apply for (4.38) the x -generalised hodograph transformation

$${}_3\mathbf{H} : \begin{cases} dx(X, T) = k^{-1}dX \\ \quad + \left\{ -3kk'V_X - \left(\frac{1}{4}(k')^2 + kk'' \right) V_X^2 - kk'V_{XX} - k^2 \right\} dT \\ dt(X, T) = dT \\ v(x, t) = X, \end{cases} \quad (4.39)$$

which transforms (4.38) to

$$\begin{aligned} V_T = & k^3 V_{XXX} + \left(\frac{3}{2}k^2 k' + 3k^3 \frac{k''}{k'} \right) V_X V_{XX} + 3k^3 V_{XX} \\ & + \left(\frac{3}{2}k^2 k'' + k^3 \frac{k'''}{k'} \right) V_X^3 + 3 \left(k^2 k' + k^3 \frac{k''}{k'} \right) V_X^2 + 2k^3 V_X. \end{aligned} \quad (4.40)$$

where $k = k(V)$. It is remarkable that eq. (4.13) with $\lambda_1 = \lambda_2 = \lambda_3 = C = 0$, i.e.

$$\tilde{V}_{\tilde{T}} = \tilde{V}_{\tilde{X}\tilde{X}\tilde{X}} - \frac{3}{4}\tilde{V}_{\tilde{X}}^{-1}\tilde{V}_{\tilde{X}\tilde{X}}^2 \quad (4.41)$$

can also be transformed to (4.40) by the x -generalised hodograph transformation

$${}_3\mathbf{H} : \begin{cases} d\tilde{X}(X, T) = k^{-1}dX \\ \quad + \left\{ -3kk'V_X - \left(\frac{1}{4}(k')^2 + kk'' \right) V_X^2 - kk'V_{XX} - k^2 \right\} dT \\ d\tilde{T}(X, T) = dT \\ \tilde{V}(\tilde{X}, \tilde{T}) = \frac{1}{2}e^{2X}. \end{cases} \quad (4.42)$$

Equation (4.41) is the potential form of the third-order equation

$$W_{\tilde{T}} = W_{\tilde{X}\tilde{X}\tilde{X}} + 3W^{-1}W_{\tilde{X}}W_{\tilde{X}\tilde{X}}, \quad (4.43)$$

which is part of the linearisable hierarchy VIII in [5]. The relation of (4.43) to (4.40) is established by

$$W_{\tilde{X}} = \tilde{V}^4, \quad (4.44)$$

whereas the linearisation of (4.43) in

$$U_{\tilde{T}} = U_{\tilde{X}\tilde{X}\tilde{X}} \quad (4.45)$$

is obtained [5] by

$$W^2 = 2U_{\tilde{X}}. \quad (4.46)$$

The composition of the above transformations for (4.38) and (4.41), that is the change of coordinates

$$(\tilde{X}, \tilde{T}, \tilde{V}(\tilde{X}, \tilde{T})) \mapsto (x, t, v(x, t)),$$

results in the transformation

$$v(x, t) = \frac{1}{2} \ln(2\tilde{V}), \quad dx(\tilde{X}, \tilde{T}) = d\tilde{X}, \quad dt(\tilde{X}, \tilde{T}) = d\tilde{T}. \quad (4.47)$$

Thus the transformation between (4.37) and (4.41), that is the coordinate transformation

$$(\tilde{X}, \tilde{T}, \tilde{V}(\tilde{X}, \tilde{T})) \mapsto (x, t, u(x, t)),$$

takes the form

$$u(x, t) = \left(\frac{\tilde{V}_{\tilde{X}}}{2\tilde{V}} \right)^{1/2}, \quad x = \tilde{X}, \quad t = \tilde{T}. \quad (4.48)$$

Finally, the linearisation of (4.37) in (4.45) is given by the following nonlocal transformation:

$$\frac{\partial U(x, t)}{\partial x} = \sqrt{2} u \exp \left(\int u^2(x, t) dx \right). \quad (4.49)$$

The recursion operator for (4.37) is nonlocal and takes the form

$$\begin{aligned} R[u] = & D_x^2 + 2u^2 D_x + 10uu_x + u^4 \\ & + 2 \left(u_{xx} + 2u^2 u_x + 2ue^{-2 \int u^2 dx} \int e^{2 \int u^2 dx} u_x^2 dx \right) D^{-1} u \\ & - 2ue^{-2 \int u^2 dx} D_x^{-1} \left[(u_{xx} + 2u^2 u_x) e^{2 \int u^2 dx} + 2u \int e^{2 \int u^2 dx} u_x^2 dx \right]. \end{aligned} \quad (4.50)$$

The recursion operator (4.50) is obtained by transforming the recursion operator of eq. (4.41), namely

$$R[\tilde{V}] = \left(D_{\tilde{X}} - D_{\tilde{X}}^{-1} \left[\tilde{V}^{-1} \tilde{V}_{\tilde{X}} + \tilde{V}^2 \right] D_{\tilde{X}} \right)^2, \quad (4.51)$$

with the transformations (4.48). Note that (4.50) may also be written in the following nonlinear, but equivalent, form:

$$\begin{aligned} R[u] = & D_x^2 + 2u^2 D_x + 10uu_x + u^4 + 2(u_{xx} + 2u^2 u_x) D_x^{-1} u \\ & - 2ue^{-2 \int u^2 dx} D_x^{-1} e^{2 \int u^2 dx} [u_{xx} + 2u^2 u_x - 2u_x^2 D_x^{-1} u]. \end{aligned} \quad (4.52)$$

To get the next equation in the hierarchy of eq. (4.37) we apply the recursion operator on the t -translation symmetry of (4.37), i.e. $R[u]u_t$, which leads to the fifth-order equation

$$\begin{aligned} u_t = & u_{xxxxx} + 5u^2 u_{xxx} + 40uu_x u_{xx} + 25u u_{xx}^2 + 50u_x^2 u_{xx} + 10u^4 u_{xxx} \\ & + 120u^3 u_x u_{xx} + 140u^2 u_x^3 + 10u^6 u_{xx} + 70u^5 u_x^2 + 5u^8 u_x. \end{aligned} \quad (4.53)$$

We recall that the coefficients I_j of the term $I_j D_x^{-1} J_j$ in the linear recursion operator (4.50) are symmetries of the given equation. Therefore, eq. (4.50) admits the following two nonlocal symmetries:

$$Z_1 = ue^{-2 \int u^2 dx} \frac{\partial}{\partial u} \quad (4.54)$$

$$Z_2 = \left(u_{xx} + 2u^2 u_x + 2ue^{-2 \int u^2 dx} \int e^{2 \int u^2 dx} u_x^2 dx \right) \frac{\partial}{\partial u} \quad (4.55)$$

We can now construct more nonlocal symmetries for (4.37) by applying the recursion operator on the nonlocal symmetries Z_1 and Z_2 . For Z_1 we get zero, but for Z_2 we obtain the fourth order nonlocal symmetry

$$\begin{aligned} Z_3 = & \left(u_{xxxx} + 4u^2 u_{xxx} + 28uu_x u_{xx} + 8u_x^3 + 6u^4 u_{xx} + 32u^3 u_x^2 + 4u^6 u_x \right. \\ & \left. - 2ue^{-2 \int u^2 dx} \int e^{2 \int u^2 dx} [u_{xx}^2 + 4u^2 u_x u_{xx} - 4uu_x^3 + 2u^4 u_x^2] dx \right) \frac{\partial}{\partial u}. \end{aligned} \quad (4.56)$$

5 Conclusion

We present recursion operators for a number of integrable equations of the semilinear form (1.1) and the quasilinear form (1.2). Those are given in Proposition 2.1 and Proposition 3.1. In addition we report the potential forms of those equations and apply a sequence of coordinate transformations, namely the hodograph transformation and the x -generalised hodograph transformation. In some cases the transformed equations admit local recursion operators, which are given explicitly. In other cases the transformed equations do not admit a recursion operator of the form (1.3) and we suspect that those equations admit only nonlocal recursion operators and/or recursion operators which depend explicitly on x and t . We do not study such recursion operators here. However, we make one exception

namely for eq. (4.37), for which we calculate the nonlocal recursion operator by using coordinate transformations.

Below we give a list of all semilinear and quasilinear equations (up to $u \rightarrow k(u)$) for which we obtained recursion operators in this paper:

Semilinear equations:

$$u_t = u_{xxx} + \lambda_1 u_x^3 + \lambda_2 u_x^2 + \lambda_3 u_x + \lambda_4$$

$$u_t = u_{xxx} + n(u)u_x u_{xx} + m(u)u_x^3 + q(u)u_x$$

$$q''' - nq'' - 3n'q' - \frac{2}{3}n^2q' + 8mq' = 0$$

$$9m' - 3n'' - 6nm + \frac{2}{3}n^3 = 0$$

$$u_t = u_{xxx} + 3n(u)u_x u_{xx} - \frac{1}{2}uu_x^3 + q(u)u_x$$

$$q''' - 3nq'' - 9\left(n' + \frac{2}{3}n^2 + \frac{4}{9}u\right)q' = 0$$

$$n'' = 2n^3 + nu - \frac{1}{2}$$

$$u_t = u_{xxx} + 3u^{-1}u_x u_{xx} + \lambda_1 u^4 u_x + \lambda_2 u^2 u_x + \lambda_3 u_x$$

$$u_t = u_{xxx} - 3u^{-1}u_x u_{xx} + \frac{3}{2}u^{-2}u_x^3 + \lambda_1 u^2 u_x + \lambda_2 u^{-2}u_x + \lambda_3 u_x$$

$$u_t = u_{xxx} - \frac{\beta^2}{8}u_x^3 + \lambda_1 e^{\beta u}u_x + \lambda_2 e^{-\beta u}u_x + \lambda_3 u_x$$

$$u_t = u_{xxx} + \frac{\beta^2}{8}u_x^3 + \lambda_1 \cos(\beta u)u_x + \lambda_2 \sin(\beta u)u_x + \lambda_3 u_x$$

$$u_t = u_{xxx} + \lambda_1 u^2 u_x + \lambda_2 u u_x + \lambda_3 u_x$$

$$u_t = u_{xxx} + 3u^2 u_{xx} + 9u u_x^2 + 3u^4 u_x$$

Quasilinear equations:

$$u_t = u^3 u_{xxx} + 3u^2 u_x u_{xx} + \lambda_1 u^3 u_x + \lambda_2 u^2 u_x + \lambda_3 u_x$$

$$u_t = u^3 u_{xxx} + \lambda_1 u^3 u_x + \lambda_2 u^{-1} u_x + \lambda_3 u_x$$

Linearisable third-order equations:

For the reader's convenience we also list here the linearisable third-order evolution equations which admit x - and t -independent recursion operators given in [5]. Below β , γ , α are arbitrary constants and k a nonconstant \mathcal{C}^3 function of u :

$$u_t = u_{xxx} + 3 \left(\frac{k''}{k'} + \alpha k' \right) u_x u_{xx} + \left(\frac{k'''}{k'} + 3\alpha k'' + \alpha^2 (k')^2 \right) u_x^3 \\ + \alpha \gamma u_x + \beta \left(u_{xx} + \left(\frac{k''}{k'} + \alpha k' \right) u_x^2 \right).$$

$$u_t = u_{xxx} + 3 \frac{k''}{k'} u_x u_{xx} + \frac{k'''}{k'} u_x^3 + \frac{3\alpha}{2} k u_{xx} + \frac{3\alpha}{2} \left(k \frac{k''}{k'} + k' \right) u_x^2 \\ + \frac{3\alpha^2}{4} k^2 u_x + \beta \left(u_{xx} + \frac{k''}{k'} u_x^2 + \alpha k u_x \right)$$

$$u_t = k^3 u_{xxx} + 3 \left(k^3 \frac{k''}{k'} - k^2 k' \right) u_x u_{xx} + \left(k^3 \frac{k'''}{k'} + 3k^2 k'' \right) u_x^3 + 3\alpha k^3 u_{xx} \\ + 3\alpha \left(k^3 \frac{k''}{k'} + k^2 k' \right) u_x^2 + 2\alpha^2 k^3 u_x.$$

The above three equations are the second equations in the linearisable hierarchies (1.8), (1.10) and (1.12), respectively. Their recursion operators are (1.9), (1.11) and (1.13), respectively.

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